Statistical modelling under epistemic data imprecision

Some results, next steps, questions,...

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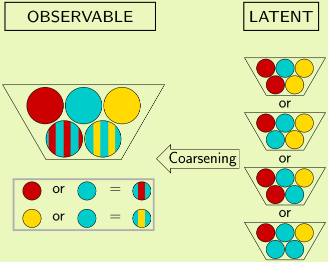
Epistemic vs. ontic interpretation (Couso, Dubois, Sánchez, 2014)

Epistemic imprecision:

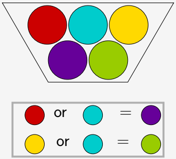
"Imprecise observation of something precise"



⇒ Truth is hidden due to the underlying coarsening mechanism

Ontic imprecision:

"Precise observation of something imprecise"

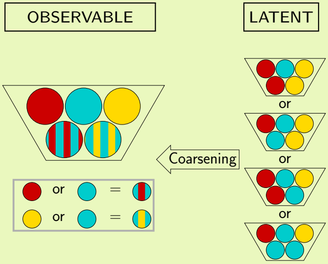


⇒ Truth is represented by coarse observation

Examples of data under epistemic imprecision

Epistemic imprecision:

"Imprecise observation of something precise"



$\Rightarrow$ Truth is hidden due to the underlying coarsening mechanism

Examples:

- Matched data sets with partially overlapping variables
- Coarsening as anonymization technique
- Missing data as special case

Here: PASS-data
$\mathcal{Y}$: income, X: UBII

$$\Omega_{\mathcal{Y}} = \{<, \geq, \text{na}\}$$
$$\Omega_X = \{0 \text{ (no)}, 1 \text{ (yes)}\}$$

Basic problem

OBSERVABLE		LATENT
$\mathcal{Y}$ coarse data		Y latent variable
		<u>Main goal:</u>
	coarsening mechanism	Estimation of
$p_{\mathscr{y}_i} = P(\mathcal{Y}_i = \mathscr{y}_i),\ i = 1, \ldots, n$	$q_{\mathscr{y}_i \vert y_i} = P(\mathcal{Y}_i = \mathscr{y}_i \vert Y_i = y_i)$	$\pi_{ij} = P(Y_i = j)$

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$p_{y_i} = P(\mathcal{Y}_i = y_i),\ i = 1, \ldots, n$

Observation model $\mathcal{Q}$

error-freeness

coarsening mechanism

$q_{y_i|y_i} = P(\mathcal{Y}_i = y_i | Y_i = y_i)$

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Y latent variable

Main goal:

Estimation of

$\pi_{ij} = P(Y_i = j)$

1. Frequentist approach:

   Find maximum-likelihood estimator of $\boldsymbol{\gamma} = (\boldsymbol{q}^T_{y_i|\boldsymbol{y_i}},\ \boldsymbol{\pi}^T_{\boldsymbol{y}})^T$

   (i.i.d. case , regression case)

2. Bayesian approach:

   Find posterior mean

   (Next step)

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Likelihood for parameters $\boldsymbol{p} = (p_1, \ldots, p_{|\Omega_{\mathcal{Y}}|-1})^T$

$L(\boldsymbol{p}) \propto \prod_{\mathscr{y} \in \Omega_{\mathcal{Y}}} p_{\mathscr{y}}^{n_{\mathscr{y}}}$ is *uniquely* maximized by

$\boxed{\hat{p}_{\mathscr{y}} = \frac{n_{\mathscr{y}}}{n},}\qquad \mathscr{y} \in \{1, \ldots, |\Omega_{\mathcal{Y}}| - 1\}$

and thus $\hat{p}_{|\Omega_{\mathcal{Y}}|} = 1 - \sum_{m=1}^{|\Omega_{\mathcal{Y}}|-1} \hat{p}_m$.

Use the **connection between $\boldsymbol{p}$ and $\boldsymbol{\gamma}$**

$$\Phi(\boldsymbol{\gamma}) = \mathbf{p}$$

and the **invariance of the likelihood** under parameter transformations, i.e.:

$$\boxed{\hat{\Gamma} = \{\boldsymbol{\gamma} \mid \Phi(\boldsymbol{\gamma}) = \hat{\boldsymbol{p}}\}}$$

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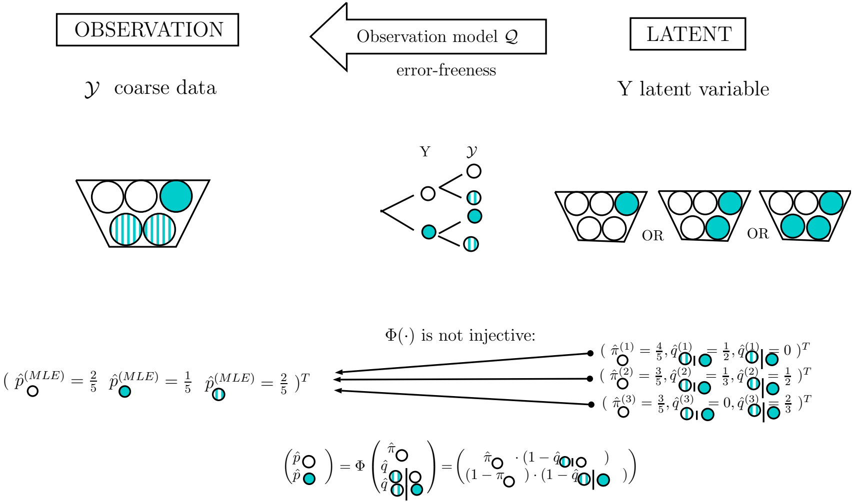

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Illustration (PASS data)

$n_< = 238,\ n_\geq = 835,\ n_{\text{na}} = 338$

$\hat{\pi}_< \in \left[\frac{238}{1411},\ \frac{238+338}{1411}\right]$

Basic idea for the regression case

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error-freeness

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coarsening mechanism

$p_{x\mathscr{y}_i} = P(\mathcal{Y}_i = \mathscr{y}_i | X = x),\ i = 1, \dots, n$
$x \in \{0,\ 1\}$

$q_{\mathscr{y}|xy} = P(\mathcal{Y} = \mathscr{y} | X = x, Y = y)$

$\pi_{ij} = P(Y_i = j | \mathbf{x}_i) = \frac{\exp(\beta_{j0} + \mathbf{x}_i^T \boldsymbol{\beta}_j)}{1 + \sum_{s=1}^{K-1} \exp(\beta_{s0} + \mathbf{x}_i^T \boldsymbol{\beta}_s)}$

$\pi_{iK} = \left(1 + \sum_{s=1}^{K-1} \exp(\beta_{s0} + \mathbf{x}_i^T \boldsymbol{\beta}_s)\right)^{-1}$

(multinomial logit model)

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For fixed x, $(n_{x1}, \dots n_{x|\Omega_\mathcal{Y}|}) \sim M(n_x, (p_{x1}, \dots, p_{x|\Omega_\mathcal{Y}|}))$ with conditional probabilities $p_{x\mathscr{y}} = P(\mathcal{Y} = \mathscr{y} | X = x)$.

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Illustration (PASS data)

$\hat{\pi}_{0<} \in [0.41,\ 0.64]$ $\hat{\pi}_{1<} \in [0.10,\ 0.34]$

$\hat{\beta}_{<0} \in [-0.37,\ 0.59]$ $\hat{\beta}_{<} \in [-1.83,\ -1.25]$

Reliable incorporation of auxiliary information

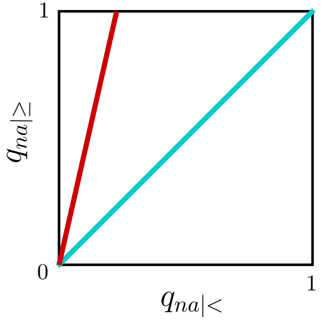
Starting from point-identifying assumptions, we use sensitivity parameters to allow inclusion of partial knowledge.

Assumption about exact value of $R = \frac{q_{na|\geq}}{q_{na|<}}$ (Nordheim, 1984):

e.g. $\mathcal{Q}$ specified by R=1, R=4

where R=1 corresponds to CAR

(Heitjan, Rubin, 1991).



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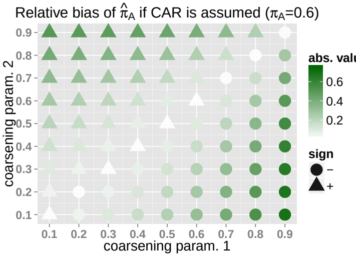

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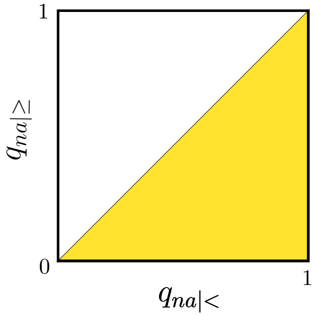
Rough evaluation of R:

e.g. $\mathcal{Q}$ specified by R $\leq$ 1:

low income group has a higher tendency

to report "na"



Sensitivity analysis (cf. Vansteelandt, Goetghebeur, Kenward, Molenberghs, 2006)

- $R = \frac{q_{na|\geq}}{q_{na|<}}$: sensitivity parameter $\delta \in \Delta$
- Honestly Estimated Ignorance Region (HEIR) $\hat{ir}(\pi_Y, \Delta)$
  - Example 1: No knowledge about R
    $\Rightarrow \hat{ir}(\pi_<, [0, \infty)) = \left[\frac{n_<}{n_< + n_\geq + n_{na}}, \ \frac{n_< + n_{na}}{n_< + n_\geq + n_{na}} \right]$
  - Example 2: $R \leq 1$
    $\Rightarrow \hat{ir}(\pi_<, [0, 1]) = \left[\frac{n_<}{n_< + n_\geq + n_{na}}, \ \frac{n_<}{n_< + n_\geq} \right]$

How to adopt for uncertainty induced by finite sampling?

Uncertainty regions...

... for the ignorance region?

... for the parameter of interest?

Likelihood

$$L(\pi_<, q_{na|<}, q_{na|\geq} || \mathbf{y}_1, \ldots, \mathbf{y}_n)$$
$$\propto ((1 - q_{na|<}) \cdot \pi_<)^{n_<} \cdot ((1 - q_{na|\geq}) \cdot (1 - \pi_<))^{n_\geq}$$
$$\cdot (q_{na|<} \cdot \pi_A + q_{na|\geq} \cdot (1 - \pi_<))^{n_{na}}$$

Likelihood ratio test:

- $H_0 : \boldsymbol{\gamma} = \gamma_0 \qquad H_1 : \boldsymbol{\gamma} \neq \gamma_0$
- $LQ = 2 log \frac{L(\hat{\gamma}_{ML})}{L(\hat{\gamma}_{0,ML})} = 2 \log(l(\hat{\gamma}_{ML}) - l(\hat{\gamma}_{0,ML})$
- Reject H_0 if $LQ \geq \chi^2_{dim(\gamma)}$

Basic problem, now: Bayesian approach

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1. Frequentist approach:

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   (i.i.d. case , regression case)

2. Bayesian approach:

   Find posterior mean

   (Next step)

Summary and outlook

- Via the observation model $\mathcal{Q}$ maximum-likelihood estimators referring to the latent variable may be obtained for both cases
  - ... the homogeneous case
  - ... the case with categorical covariates
- Proper inclusion of auxiliary information via further restrictions on $\mathcal{Q}$

Next steps:

- Likelihood-based hypothesis tests and uncertainty regions for coarse categorical data
- Inclusion of auxiliary information via sets of priors
- Consideration of other "deficiency" processes

References

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