Reliable likelihood inference under coarse data and the (non-)testability of coarsening assumptions

Leibniz-Institut für Bildungsverläufe
Arbeitsbereich: Methoden der Survey-Statistik

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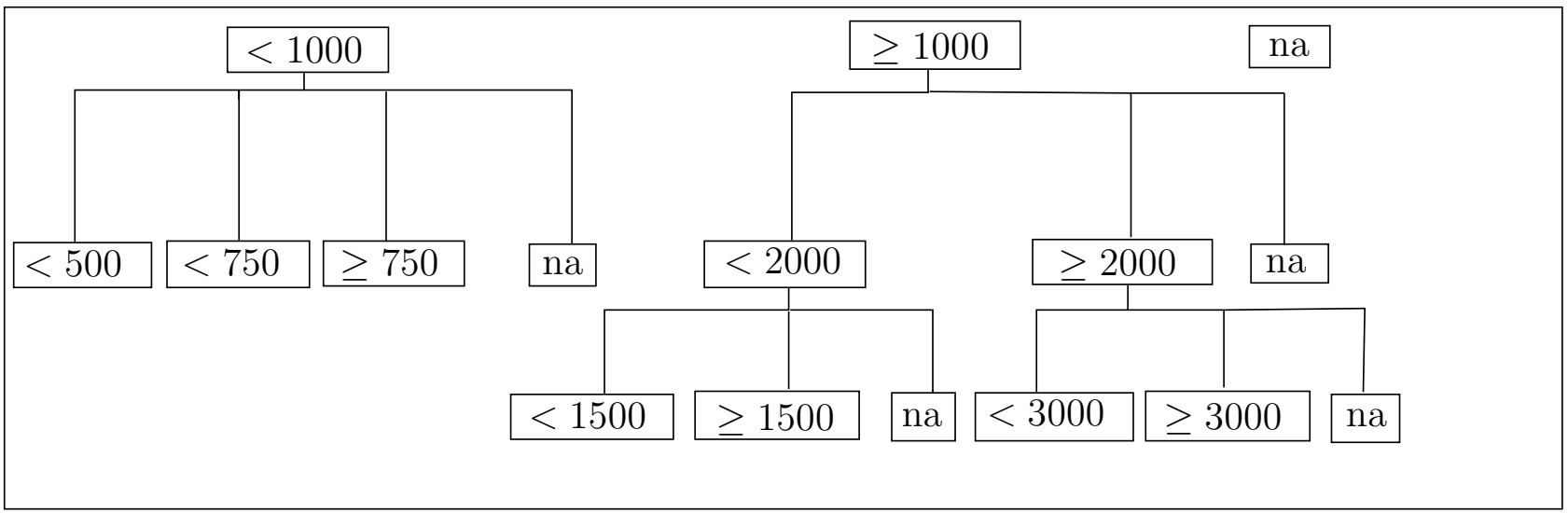
Example for coarse categorical data

- coarse data: Data that can not be observed in the required accuracy originally intended in the substance matter context
- German Panel Study "Labour Market and Social Security" conducted by the IAB (PASS study)
- sensitive income question precisely asked, follow-up questions are directed to nonresponders:



- missing data as a special case

What's the problem?

Common dealing with incomplete data: assumptions

$\Rightarrow$ Missing at random (MAR) / coarsening at random (CAR)

$\Rightarrow$ Frequently: assumptions only for pragmatic reasons

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Here:

1.) Reliable likelihood inference

2.) Testability of coarsening assumptions

3.) Reliable regression estimators

Coarse data (categorical setting)

LATENT

i.i.d. random sample $(x_1, y_1), \ldots, (x_n, y_n)$ of categ. variables (X,Y) with $\Omega_X \times \Omega_Y$

coarsening

$y_i \in \mathcal{Y}_i$, i=1,...,n

OBSERVABLE

i.i.d. random sample $(x_1, \mathcal{Y}_1), \ldots, (x_n, \mathcal{Y}_n)$ of categ. variables (X,$\mathcal{Y}$) with $\Omega_X \times \Omega_{\mathcal{Y}}$, $\Omega_{\mathcal{Y}} = \mathcal{P}(\Omega_Y) \setminus \{\emptyset\}$

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		$\mathcal{Y}$							
		{a}	{b}	{c}	{a,b}	{a,c}	{b,c}	{a,b,c}	sum
X	0	$n_{0\{a\}}$	$n_{0\{b\}}$	$n_{0\{c\}}$	$n_{0\{a,b\}}$	$n_{0\{a,c\}}$	$n_{0\{b,c\}}$	$n_{0\{a,b,c\}}$	n_0
	1	$n_{1\{a\}}$	$n_{1\{b\}}$	$n_{1\{c\}}$	$n_{1\{a,b\}}$	$n_{1\{a,c\}}$	$n_{1\{b,c\}}$	$n_{1\{a,b,c\}}$	n_1

Contingency table for coarse data

Coarse data (categorical setting)

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		$\mathcal{Y}$			
		{a}	{b}	{a,b}	sum
X	0	$n_{0\{a\}}$	$n_{0\{b\}}$	$n_{0\{a,b\}}$	n_0
	1	$n_{1\{a\}}$	$n_{1\{b\}}$	$n_{1\{a,b\}}$	n_1

Contingency table for missing data

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- **UBII**: receipt of Unemployment Benefit II
- $\mathcal{Y}$: categorical income
  - $\{<\}$: $< 1000€$
  - $\{\geq\}$: $\geq 1000€$
  - $\{<, \geq\}$: $<$ or $\geq$

UBII

	$\mathcal{Y}$			
	$\{<\}$	$\{\geq\}$	$\{<, \geq\}$	sum
0	38	385	95	518
1	36	42	9	87

PASS, w5 (Trappmann, 2010)

Coarse data (categorical setting)

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		$\mathcal{Y}$		
		$<$	$\geq$	sum
UBII	0	38+**95**	385	518
	1	36+**9**	42	87

potential true table

		$\mathcal{Y}$			
		$\{<\}$	$\{\geq\}$	$\{<, \geq\}$	sum
UBII	0	38	385	**95**	518
	1	36	42	**9**	87

observed table

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		$\mathcal{Y}$		
		$<$	$\geq$	sum
UBII	0	38	385+**95**	518
	1	36	42+**9**	87

potential true table

		$\mathcal{Y}$			
		$\{<\}$	$\{\geq\}$	$\{<, \geq\}$	sum
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		$\mathcal{Y}$		
		$<$	$\geq$	sum
UBII	0	38+**20**	385+**75**	518
	1	36+**5**	42+**4**	87

potential true table

		$\mathcal{Y}$			
		$\{<\}$	$\{\geq\}$	$\{<, \geq\}$	sum
UBII	0	38	385	**95**	518
	1	36	42	**9**	87

observed table

General ML estimation (Plass, Augustin, Cattaneo, Schollmeyer, 2015)

LATENT

$\pi_{xy} := P(Y=y|X=x)$

(error-freeness)

Observation model

$q_{y|xy} := P(\mathcal{Y} = y|X=x, Y=y)$

OBSERVABLE

$p_{xy} := P(\mathcal{Y} = y|X=x)$

General ML estimation (Plass, Augustin, Cattaneo, Schollmeyer, 2015)

LATENT

$$\vartheta = (\pi_{xy}^{\mathsf{T}},\ \mathsf{q}_{y|xy}^{\mathsf{T}})^{\mathsf{T}}$$

OBSERVABLE

$$\mathsf{p}_{xy} := \mathsf{P}(\mathcal{Y} = y|\mathsf{X}{=}\mathsf{x})$$

General ML estimation (Plass, Augustin, Cattaneo, Schollmeyer, 2015)

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OBSERVABLE

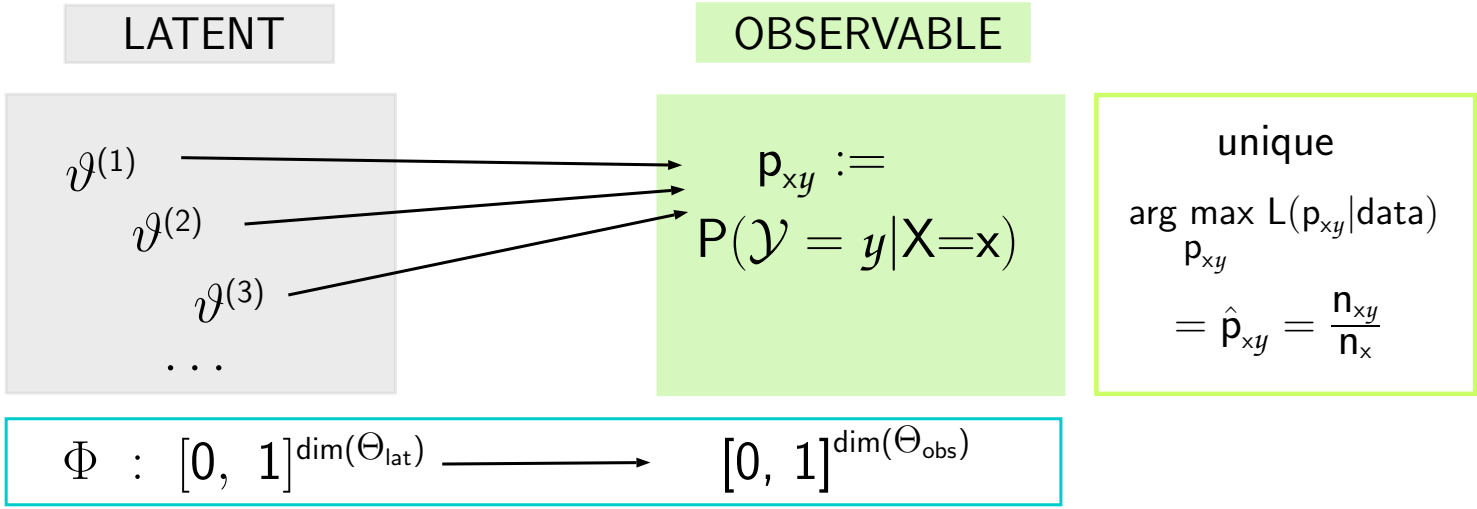
$$\mathsf{p}_{xy} := \mathsf{P}(\mathcal{Y} = y|\mathsf{X}{=}\mathsf{x})$$

unique

$$\underset{\mathsf{p}_{xy}}{\arg\max}\ \mathsf{L}(\mathsf{p}_{xy}|\mathsf{data}) = \hat{\mathsf{p}}_{xy} = \frac{\mathsf{n}_{xy}}{\mathsf{n}_{x}}$$

1.) Determine MLE of observed variable distribution

General ML estimation (Plass, Augustin, Cattaneo, Schollmeyer, 2015)



1.) Determine MLE of observed variable distribution

2.) Use connection between both worlds

$$p_{xy} = \sum_{y \in y} \left(\pi_{xy} \cdot q_{y|xy} \right).$$

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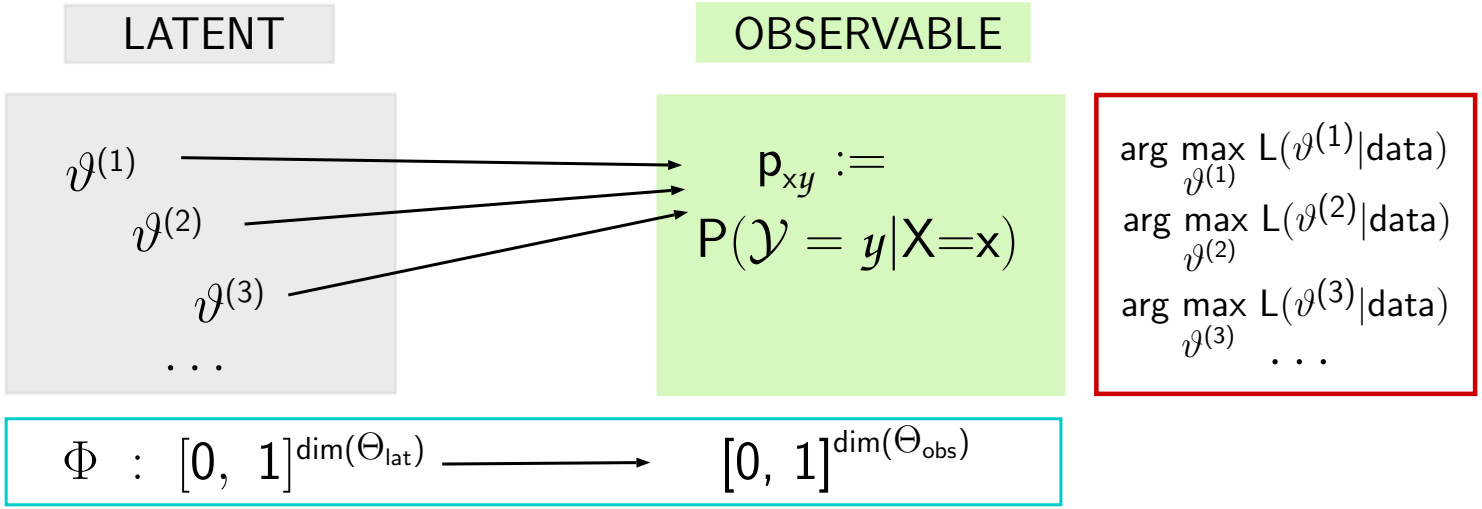

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3.) Use invariance of the likelihood

$$\hat{\pi}_{xy} \in \left[\frac{n_{x\{y\}}}{n_x}, \frac{\sum_{y \ni y} n_{xy}}{n_x} \right], \quad \hat{q}_{y|xy} \in \left[0, \frac{n_{xy}}{n_{x\{y\}} + n_{xy}} \right].$$

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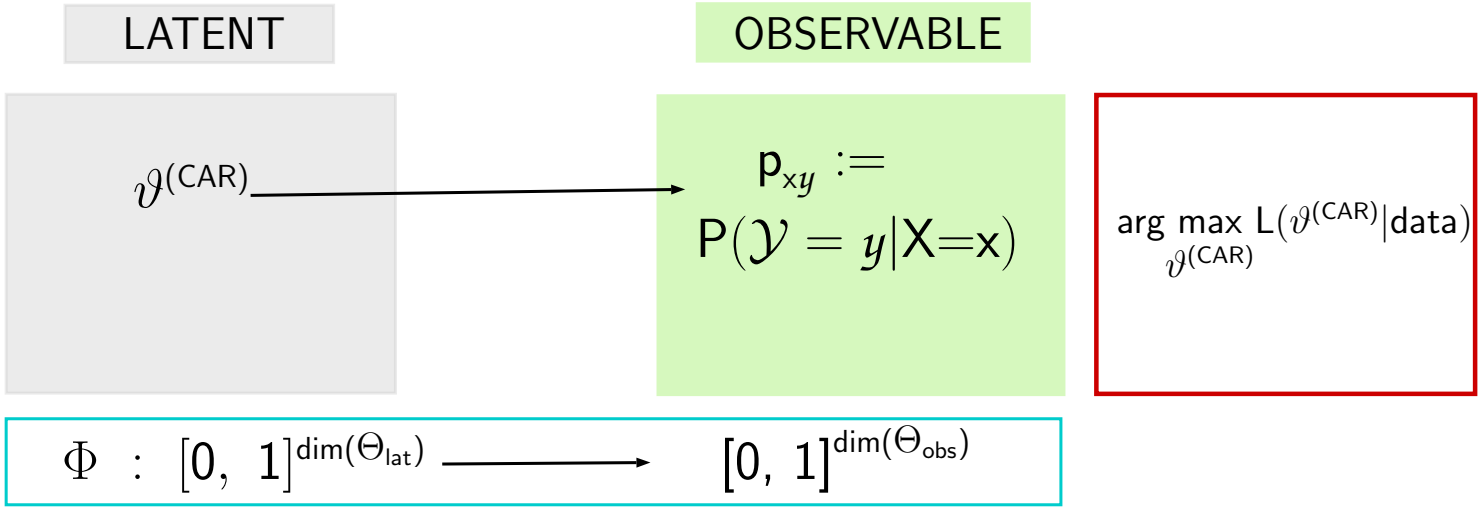

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Illustrated by the example:

- The probability of $\{<, \geq\}$ is taken to be independent of the true income category in both subgroups split by UBII:

$$q_{\{<,\geq\}|0<} = q_{\{<,\geq\}|0\geq} \quad \text{and} \quad q_{\{<,\geq\}|1<} = q_{\{<,\geq\}|1\geq}$$

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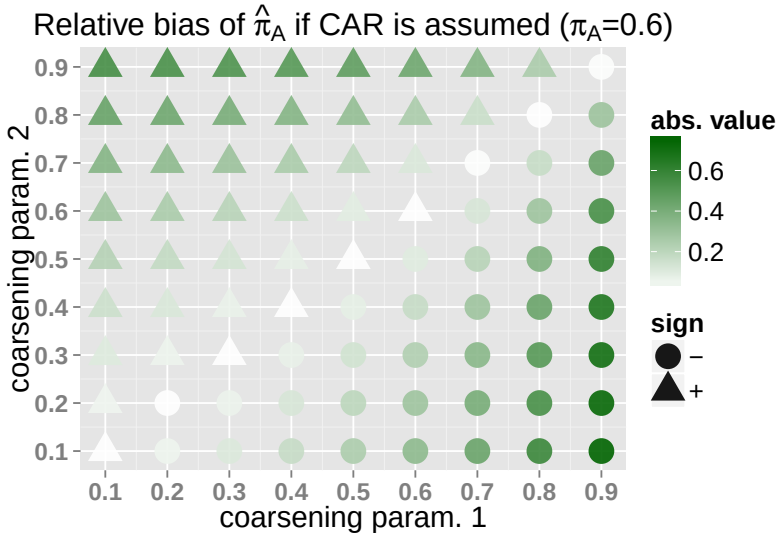

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- Resulting estimators:

$$\hat{\pi}^{(CAR)}_{x<} = \frac{n_{x\{<\}}}{n_{x\{<\}} + n_{x\{\geq\}}}, \quad \hat{q}^{(CAR)}_{\{<,\geq\}|x<} = \hat{q}^{(CAR)}_{\{<,\geq\}|x\geq} = \frac{n_{x\{<,\geq\}}}{n_x}$$

Summary: Reliable likelihood inference

Estimators for subgroup 0 (PASS data example) under ...

- ... no assumptions

$$\hat{\pi}_{0<} \in [0.07, 0.26], \quad \hat{q}_{\{<,\geq\}|0<} \in [0, 0.71], \quad \hat{q}_{\{<,\geq\}|0\geq} \in [0, 0.20]$$

- ... CAR

$$\hat{\pi}^{(CAR)}_{0<} = 0.09, \qquad \hat{q}^{(CAR)}_{\{<,\geq\}|0<} = 0.18, \qquad \hat{q}^{(CAR)}_{\{<,\geq\}|0\geq} = 0.18$$

- all tenable auxiliary information formalized via (Nordheim, 1984)

$$R_{0,<,\geq,\{<,\geq\}} = \frac{q_{\{<,\geq\}|0<}}{q_{\{<,\geq\}|0\geq}} \quad \in [0, 1] \ :$$

$$\hat{\pi}_{0<} \in [0.07, 0.09], \quad \hat{q}_{\{<,\geq\}|0<} \in [0, 0.18], \quad \hat{q}_{\{<,\geq\}|0\geq} \in [0, 0.18]$$

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Here:

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Subgroup independence (SI)

Subgroup independence (Plass, Augustin, Cattaneo, Schollmeyer, 2015):
For each fixed y and y $\in y$, $q_{y|xy}$ takes the same value $\forall x \in \Omega_X$.

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- The probability of $\{<, \geq\}$ is taken to be independent of the receipt of the UBII given y:

$$q_{\{<,\geq\}|\mathbf{0}<} = q_{\{<,\geq\}|\mathbf{1}<} \quad \text{and} \quad q_{\{<,\geq\}|\mathbf{0}\geq} = q_{\{<,\geq\}|\mathbf{1}\geq}$$

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Illustrated by the example:

- The probability of $\{<, \geq\}$ is taken to be independent of the receipt of the UBII given y:

  $q_{\{<,\geq\}|\mathbf{0}<} = q_{\{<,\geq\}|\mathbf{1}<}$ and $q_{\{<,\geq\}|\mathbf{0}\geq} = q_{\{<,\geq\}|\mathbf{1}\geq}$

- Resulting estimators (if well-defined and inside [0,1]):

$$\hat{\pi}^{(SI)}_{x<} = \frac{n_{x\{<\}}}{n_x}\frac{v}{w}, \quad \hat{q}^{(SI)}_{\{<,\geq\}|x<} = 1 - \frac{w}{v}, \quad \hat{q}^{(SI)}_{\{<,\geq\}|x\geq} = 1 - \frac{w}{z}$$

with $v = n_0 n_{1\{\geq\}} - n_{0\{\geq\}} n_1$, $w = n_{0\{<\}} n_{1\{\geq\}} - n_{0\{\geq\}} n_{1\{<\}}$
and $z = n_{0\{<\}} n_1 - n_{1\{<\}} n_0$

Testing SI? $\Rightarrow$:-)

Estimators for subgroup 0:

$$\hat{\pi}_{0<} \in [0.07, 0.26], \quad \hat{q}_{\{<,\geq\}|0<} \in [0,\ 0.71], \quad \hat{q}_{\{<,\geq\}|0\geq} \in [0,\ 0.198]$$

$$\hat{\pi}_{0<}^{(SI)} = 0.42, \qquad \hat{q}_{\{<,\geq\}|0<}^{(SI)} = -0.04, \qquad \hat{q}_{\{<,\geq\}|0\geq}^{(SI)} = 0.20$$

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$\Rightarrow$ There are data situations that might hint to (partial) incompatibility with SI

$\Rightarrow$ SI is testable in our setting

Testability of SI

- Number of degrees of freedom under SI:

$$\begin{aligned} \mathrm{df}^{\mathrm{SI}} &= \dim(\Theta_{\mathrm{obs}}) - \dim(\Theta_{\mathrm{lat,\ SI}}) \\ &= k \cdot (2^m - 2) - [k \cdot (m-1) + m \cdot (2^{m-1} - 1)] \end{aligned}$$

  with k as the number of subgroups and $m = |\Omega_Y|$

- Point-identification and testability are only valid if sufficient subgroups are available inducing $\mathrm{df} \geq 0$

m	2	3,4,5	6,7	8,9	...	20	...	50	...
minimum k	2	3	4	5	...	11	...	26	...

LR-Test for SI: Hypotheses and test statistic

- Hypotheses:

$$
\begin{aligned}
H_0 &: \quad q_{y|xy} = q_{y|x'y} \text{ for all } y \in \Omega_{\mathcal{Y}},\ x,\ x' \in \Omega_X,\ y \in \Omega_Y, \\
H_1 &: \quad q_{y|xy} \neq q_{y|x'y} \quad \text{for some } y \in \Omega_{\mathcal{Y}}\ x,\ x' \in \Omega_X,\ y \in \Omega_Y.
\end{aligned}
$$

LR-Test for SI: Hypotheses and test statistic

- Hypotheses:

$$\begin{aligned} H_0 &: q_{y|xy} = q_{y|x'y} \text{ for all } y \in \Omega_Y,\ x,\ x' \in \Omega_X,\ y \in \Omega_Y, \\ H_1 &: q_{y|xy} \neq q_{y|x'y} \text{ for some } y \in \Omega_Y\ x,\ x' \in \Omega_X,\ y \in \Omega_Y. \end{aligned}$$

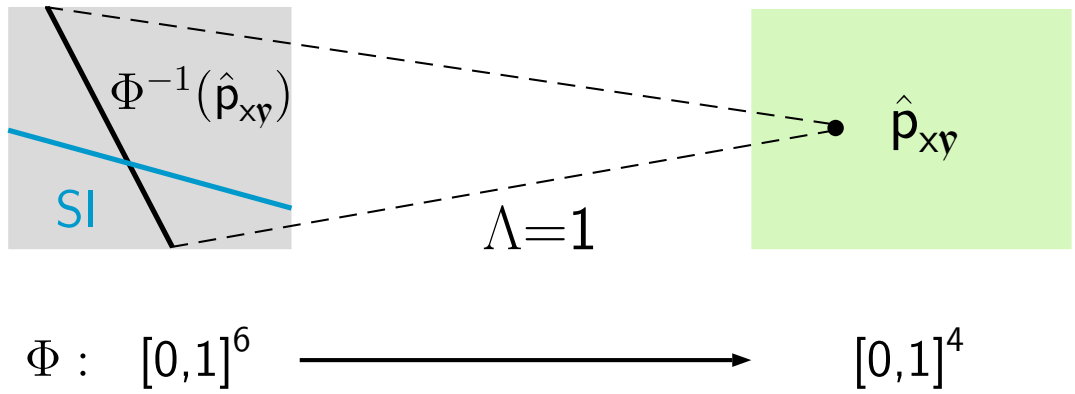
- Test based on test statistic (Wilks, 1938)

$$T = -2 \cdot \ln(\Lambda(y_1, \ldots, y_n, x_1, \ldots, x_n)) \quad \text{with}$$

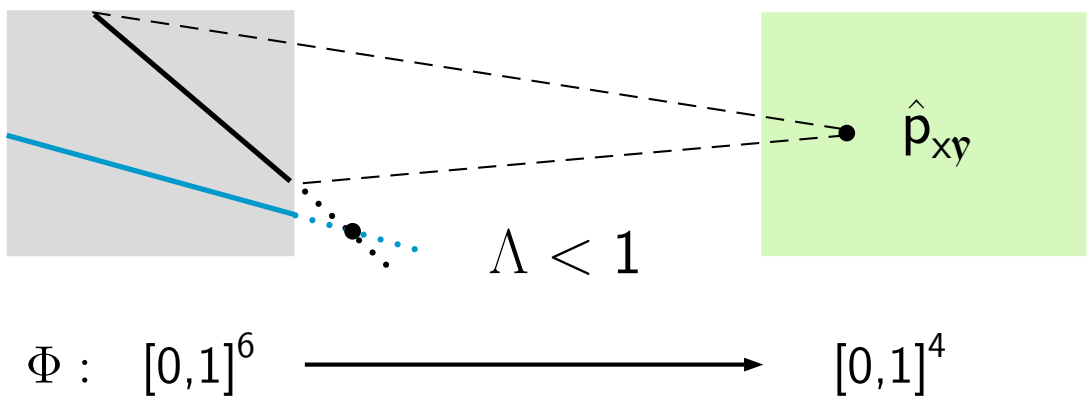
$$\Lambda(y_1, \ldots, y_n, x_1, \ldots, x_n) = \frac{\sup_{H_0} L(\vartheta | y_1, \ldots, y_n, x_1, \ldots, x_n)}{\sup_{H_0 \cup H_1} L(\vartheta | y_1, \ldots, y_n, x_1, \ldots, x_n)} .$$

LR-Test for SI: Sensitivity of Λ

- No evidence to reject SI



- Some evidence to reject SI



LR-Test for SI: Distribution of the test statistic under H_0...

- Asymptotic distribution of T under H_0:
  - $\frac{1}{2}\delta_0 + \frac{1}{2}\chi^2_1$ if $df^{SI} = 0$
  - $\chi^2_{df^{SI}}$ if $df^{SI} > 0$,

  where δ_0 is the Dirac distribution at 0 (Chernoff, 1954)
- Critical values for test decision:
  - $\chi^2_{1,1-2\cdot\alpha}$ if $df^{SI} = 0$
  - $\chi^2_{df^{SI},1-\alpha}$ if $df^{SI} > 0$,

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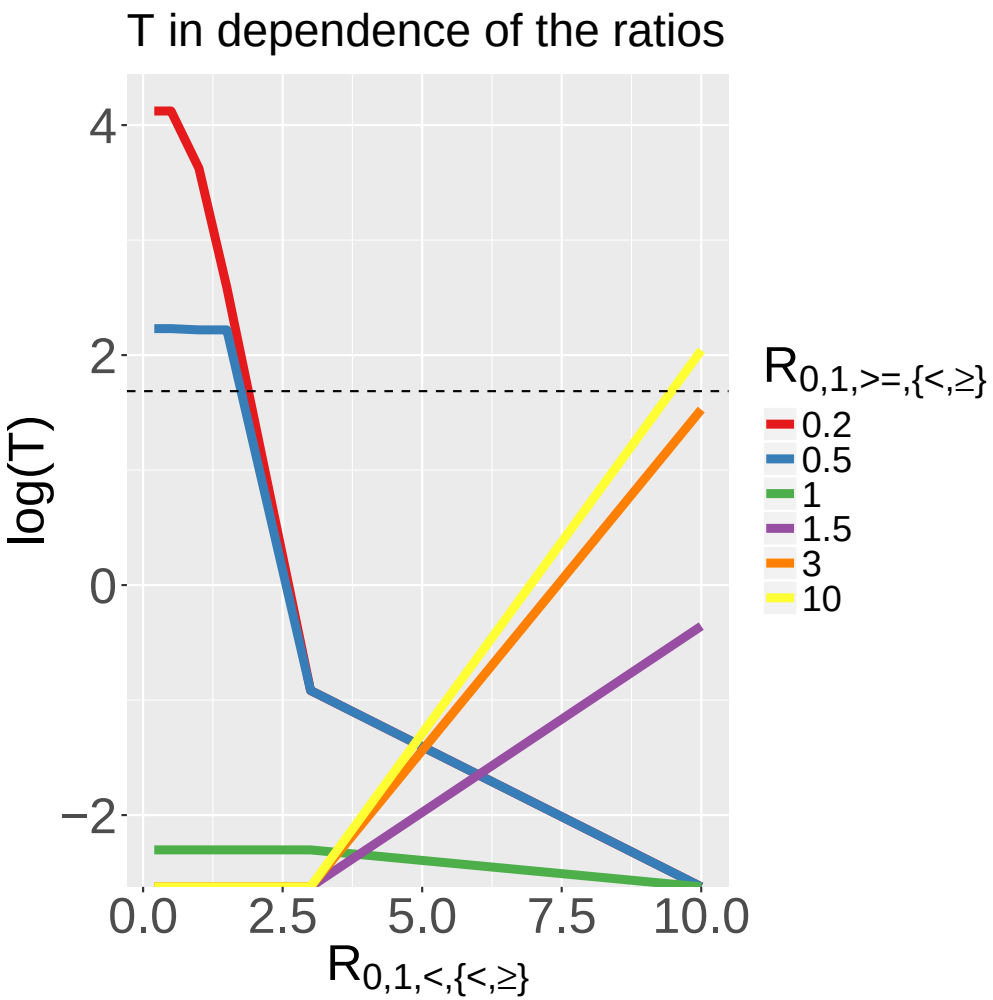
- Critical values for test decision:
  - $\chi^2_{1,1-2\cdot\alpha}$ if $df^{SI} = 0$
  - $\chi^2_{df^{SI},1-\alpha}$ if $df^{SI} > 0$,
- In data example (with $\alpha = 0.01$):

  $$T = 0.14 < 5.4 = \chi^2_{1,1-2\cdot 0.01} \quad \Rightarrow H_0 \text{ can not be rejected}$$

LR-Test for generalized SI



with ratios

$$R_{0,1,<,\{<,\geq\}} = \frac{q_{\{<,\geq\}|0<}}{q_{\{<,\geq\}|1<}}$$

and

$$R_{0,1,\geq,\{<,\geq\}} = \frac{q_{\{<,\geq\}|0\geq}}{q_{\{<,\geq\}|1\geq}}$$

Summary: CAR versus SI

	CAR	SI
Point-identifying?	always	in specific settings
Testability	generally impossible	possible in specific settings

Construction of a hypothesis test:

- H_0: (g)SI, H_1: no (g)SI
- Test statistic based on the Likelihood Ratio

(Relative) profile log-likelihood

- global log-likelihood

$$l(\Phi^{-1}(p_{00<}, p_{00\geq}, \ldots)) = l(\pi_{00<}, q_{na|00<}, q_{na|00\geq}, \ldots)$$

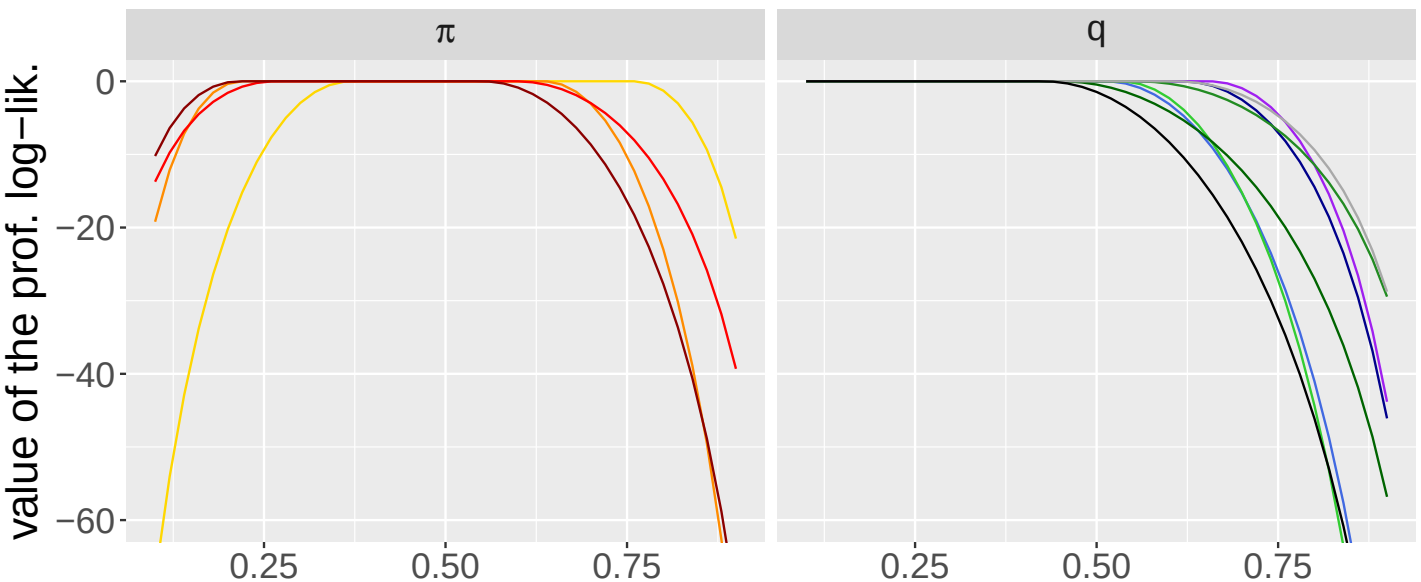
(Relative) profile log-likelihood

- global log-likelihood

$$l(\Phi^{-1}(p_{00<}, p_{00\geq}, \ldots)) = l(\pi_{00<}, q_{na|00<}, q_{na|00\geq}, \ldots)$$

- profile log-likelihood with nuissance param. $\xi = \vartheta \setminus \pi_{00<}$ or $\xi = \vartheta \setminus q_{na|00<}$

$$l(\pi_{00<}) = \max_{\xi} l(\pi_{00<}, \xi) \quad \text{or} \quad l(q_{y|\mathbf{x}y}) = \max_{\xi} l(q_{na|00<}, \xi)$$



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Reliable regression estimators (Direct method)

Logit model with linear predictor
$\eta_{\mathbf{x}} = \beta_0 + \beta_1 \cdot x_{Abitur} + \beta_2 \cdot x_{age} + \beta_{12} \cdot x_{Abitur} x_{age}$:

Response function h *Link function $g = h^{-1}$*

$$\pi_{\mathbf{x}<} = h(\eta_{\mathbf{x}}) = \frac{\exp(\eta_{\mathbf{x}})}{1 + \exp(\eta_{\mathbf{x}})} \quad \text{and} \quad \eta_{\mathbf{x}} = g(\pi_{\mathbf{x}<}) = \ln\left(\frac{\pi_{\mathbf{x}<}}{1 - \pi_{\mathbf{x}<}}\right).$$

Reliable regression estimators (Direct method)

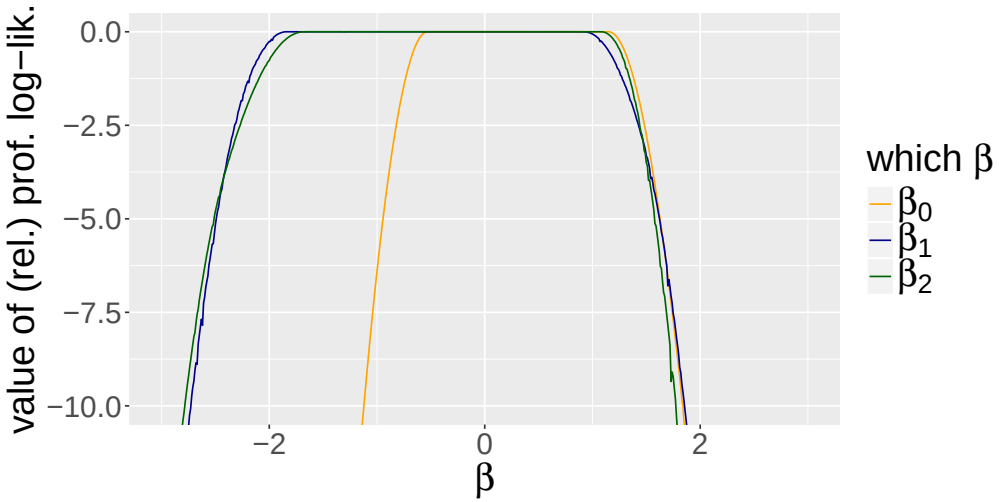
Logit model with linear predictor
$\eta_{\mathbf{x}} = \beta_0 + \beta_1 \cdot x_{Abitur} + \beta_2 \cdot x_{age} + \beta_{12} \cdot x_{Abitur} x_{age}$:

Response function h *Link function* $g = h^{-1}$

$$\pi_{\mathbf{x}<} = h(\eta_{\mathbf{x}}) = \frac{\exp(\eta_{\mathbf{x}})}{1 + \exp(\eta_{\mathbf{x}})} \quad \text{and} \quad \eta_{\mathbf{x}} = g(\pi_{\mathbf{x}<}) = \ln\left(\frac{\pi_{\mathbf{x}<}}{1 - \pi_{\mathbf{x}<}}\right).$$

$\Rightarrow$ global likelihood
$l(\beta_0, \beta_1, \ldots, q_{na|00<}, \ldots)$

$\Rightarrow$ Consider (rel.)
profile log-lik. for
each β:



Reliable regression estimators (Two-step method)

No parametric assumption on the regression model:

- Saturated model, all interactions included
- $\ln\left(\frac{\pi_{\mathbf{x}<}}{1-\pi_{\mathbf{x}<}}\right) = \beta_0 + \beta_1 \cdot x_{Abitur} + \beta_2 \cdot x_{age} + \beta_{12} x_{Abitur} x_{age}$
- $\pi_{\mathbf{x}<}$ and regression coefficients basically represent same information

Parametric assumption on the regression model:

- (some) interactions are equal to 0 (e.g. $\beta_{12} = 0$)
- $\ln\left(\frac{\pi^*_{\mathbf{x}<}}{1-\pi^*_{\mathbf{x}<}}\right) = \beta_0 + \beta_1 \cdot x_{Abitur} + \beta_2 \cdot x_{age}$
- $\pi_{\mathbf{x}<}$ and regression coefficients do not represent same information

Aim:
We want to study the impact of the parametric assumption on the regression model in the coarse data problem

Reliable regression estimators (Two-step method)

Two-step method:

1.) Estimate the bounds $\underline{\hat{\pi}}_{\mathbf{x}<}$ and $\overline{\hat{\pi}}_{\mathbf{x}<}$ (no param. assump.)

2.) Use these bounds to obtain the reliable regression estimators

No parametric assumption:

Function h is bijective $\Rightarrow$ transform $\underline{\hat{\pi}}_{\mathbf{x}<}$ and $\overline{\hat{\pi}}_{\mathbf{x}<}$ via link function

Parametric assumption: solve an optimization problem

Reliable regression estimators (Two-step method)

$\beta_1 \to \min/\max$ given

$$\underline{\hat{\pi}}_{00<} \leq \frac{\exp(\beta_0)}{1+\exp(\beta_0)} \leq \overline{\hat{\pi}}_{00<}, \qquad \underline{\hat{\pi}}_{10<} \leq \frac{\exp(\beta_0+\beta_1)}{1+\exp(\beta_0+\beta_1)} \leq \overline{\hat{\pi}}_{10<},$$

$$\underline{\hat{\pi}}_{01<} \leq \frac{\exp(\beta_0+\beta_2)}{1+\exp(\beta_0+\beta_2)} \leq \overline{\hat{\pi}}_{01<}, \qquad \underline{\hat{\pi}}_{11<} \leq \frac{\exp(\beta_0+\beta_1+\beta_2)}{1+\exp(\beta_0+\beta_1+\beta_2)} \leq \overline{\hat{\pi}}_{11<}.$$

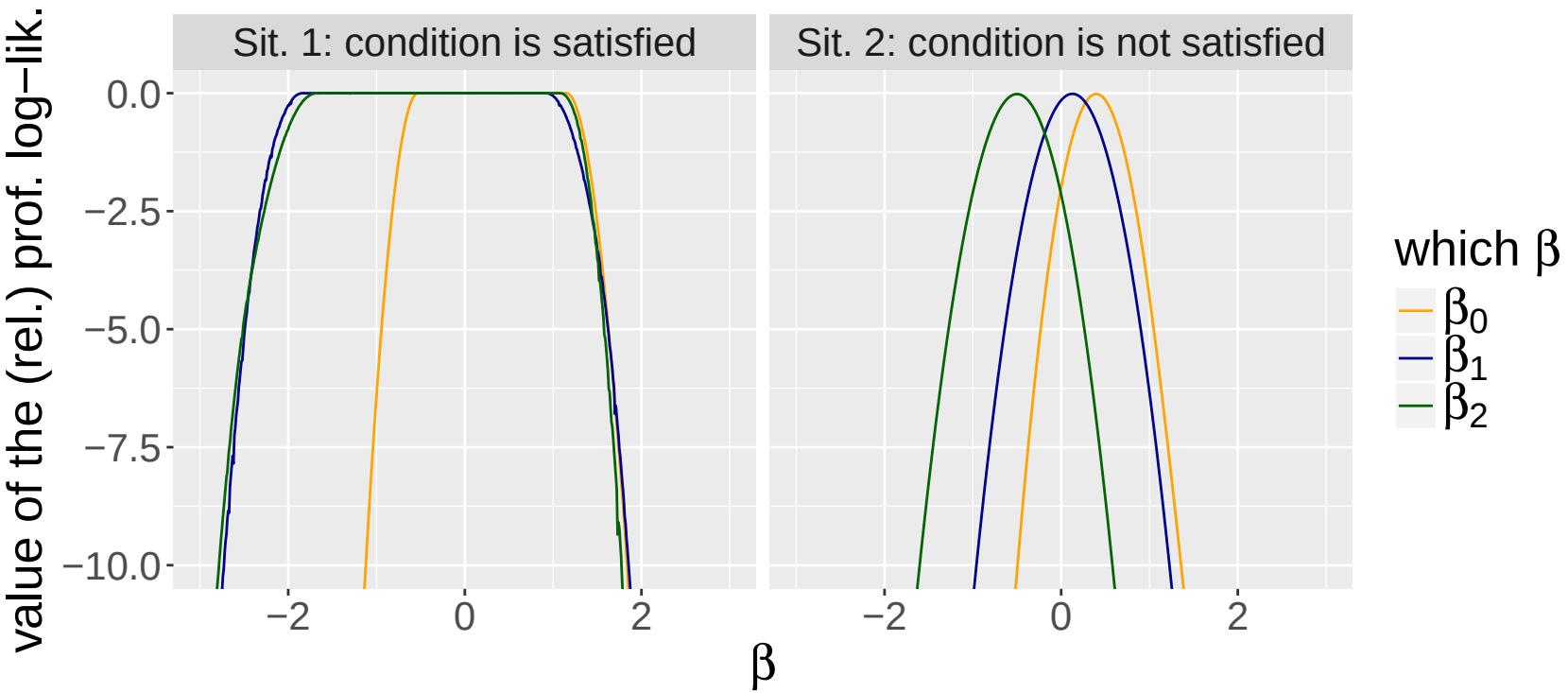
Different types of results

1.) There is a solution.
   a) Regression estimators obtainable that produce $\underline{\hat{\pi}}_{\mathbf{x}<}$ and $\overline{\hat{\pi}}_{\mathbf{x}<}$
   $\Rightarrow$ No impact on coarsening (same $\underline{\hat{q}}_{\{<,\geq\}|\mathbf{x}<}$ and $\overline{\hat{q}}_{\{<,\geq\}|\mathbf{x}<}$)
   b) The resulting regression estimators can only represent tighter bounds of the estimated latent variable distribution
   $\Rightarrow$ Tighter bounds compared to $\underline{\hat{q}}_{\{<,\geq\}|\mathbf{x}<}$ and $\overline{\hat{q}}_{\{<,\geq\}|\mathbf{x}<}$

2.) There is no solution.

Impact on the compatible coarsening scenarios?

Relative profile log-likelihood for the regression coefficients:



Results and conclusion

Reliable estimation and estimation under CAR (MAR)

- No parametric assumption on the regression model

$\hat{\beta}_0 \in [-0.53,\ 1.15]$	$\hat{\beta}_1 \in [-2.16,\ 0.92]$	$\hat{\beta}_2 \in [-2.42,\ 1.08]$	$\hat{\beta}_{12} \in [-2.76,\ 3.64]$
$\hat{\beta}_0 = 0.43$	$\hat{\beta}_1 = -0.85$	$\hat{\beta}_2 = -0.94$	$\hat{\beta}_{12} = 0.63$

- Parametric assumption on the regression model

$\hat{\beta}_0 \in [-0.53,\ 1.15]$	$\hat{\beta}_1 \in [-1.84,\ 0.92]$	$\hat{\beta}_2 \in [-1.68,\ 1.08]$
$\hat{\beta}_0 = 0.35$	$\hat{\beta}_1 = 0.05$	$\hat{\beta}_2 = 0.00$

Conclusions:

- Two methods for a reliable regression estimators have been studied
- Principally different impact on the estimated coarsening parameters
- Compatibility with observed data can be “repaired” in the presence of coarse data

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