Testing of coarsening mechanisms: Coarsening at random versus subgroup independence

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Julia Plass*, Marco Cattaneo**,
Georg Schollmeyer*, Thomas Augustin*

*Department of Statistics, Ludwig-Maximilians University and
**School of Mathematics and Physical Sciences, University of Hull

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What's the problem?

Common dealing with incomplete data: assumptions

$\Rightarrow$ Missing at random (MAR) / coarsening at random (CAR)

$\Rightarrow$ Frequently: assumptions only for pragmatic reasons

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What are we doing?

Two types of uninformative coarsening:

The coarsening is independent of ...

- ... the true underlying value (CAR)
- ... the covariate's value (subgroup independence (SI))

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The coarsening is independent of ...

- ... the true underlying value (CAR)
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Here: categorical setting

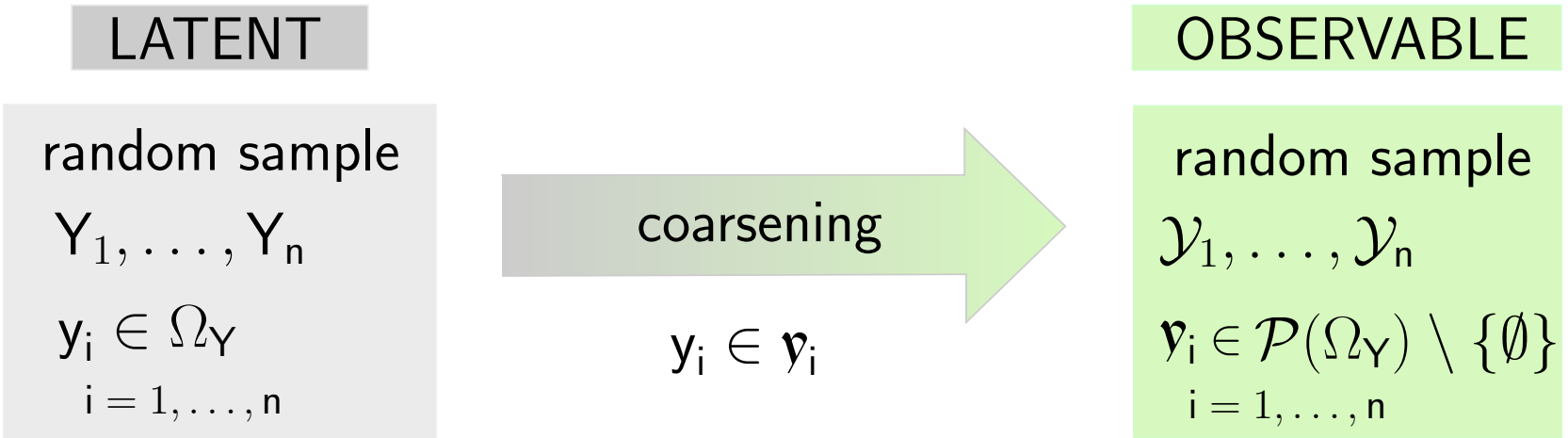
1.) No assumptions about the coarsening

2.) CAR: intuitive insight why not testable

3.) SI: construction of Likelihood Ratio test

Coarse data (categorical setting)

Epistemic interpretation of coarse data (Couso, Dubois, 2014):



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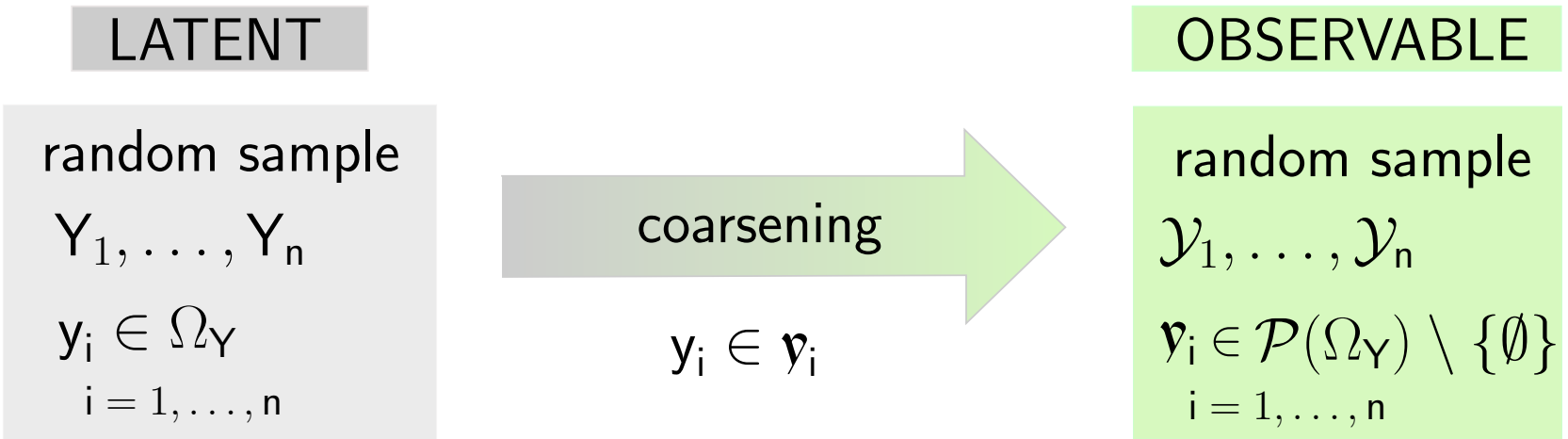

		$\mathcal{Y}$							
		{a}	{b}	{c}	{a,b}	{a,c}	{b,c}	{a,b,c}	sum
X	0	$n_{0\{a\}}$	$n_{0\{b\}}$	$n_{0\{c\}}$	$n_{0\{a,b\}}$	$n_{0\{a,c\}}$	$n_{0\{b,c\}}$	$n_{0\{a,b,c\}}$	n_0
	1	$n_{1\{a\}}$	$n_{1\{b\}}$	$n_{1\{c\}}$	$n_{1\{a,b\}}$	$n_{1\{a,c\}}$	$n_{1\{b,c\}}$	$n_{1\{a,b,c\}}$	n_1

Table: Contingency table for coarse data

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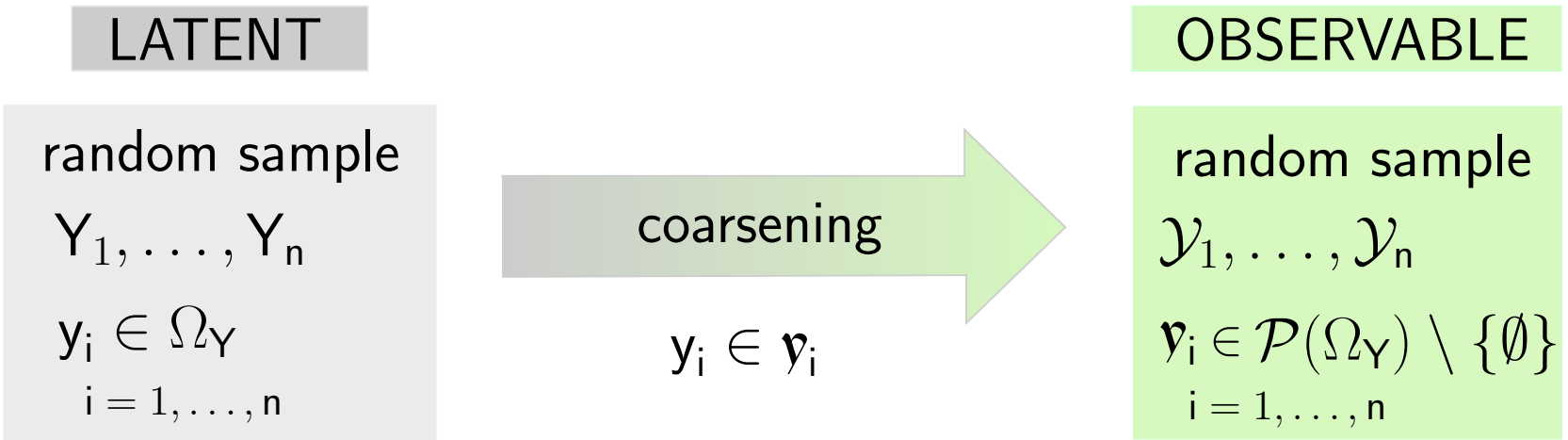

		$\mathcal{Y}$			
		{a}	{b}	{a,b}	sum
X	0	$n_{0\{a\}}$	$n_{0\{b\}}$	$n_{0\{a,b\}}$	n_0
	1	$n_{1\{a\}}$	$n_{1\{b\}}$	$n_{1\{a,b\}}$	n_1

Table: Contingency table for missing data

Coarse data (categorical setting)

Epistemic interpretation of coarse data (Couso, Dubois, 2014):

LATENT

random sample

$Y_1, \ldots, Y_n$

$y_i \in \Omega_Y$

$i = 1, \ldots, n$

coarsening

$y_i \in \mathcal{Y}_i$

OBSERVABLE

random sample

$\mathcal{Y}_1, \ldots, \mathcal{Y}_n$

$\mathcal{Y}_i \in \mathcal{P}(\Omega_Y) \setminus \{\emptyset\}$

$i = 1, \ldots, n$

- **UBII**: receipt of Unemployment Benefit II
- $\mathcal{Y}$: categorical income
  - {a}: $< 1000€$
  - {b}: $\geq 1000€$
  - {a,b}: $<$ or $\geq$

UBII	$\mathcal{Y}$ {a}	{b}	{a,b}	sum
0	38	385	95	518
1	36	42	9	87

Table: PASS, w5 (Trappmann, 2010)

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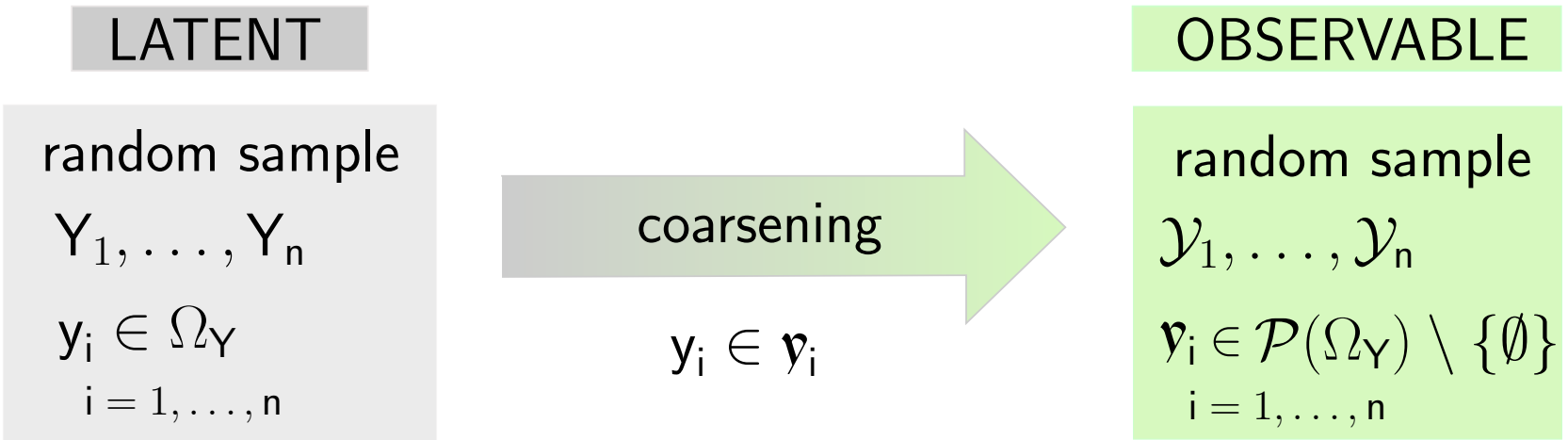

		$\mathcal{Y}$		
		a	b	sum
UBII	0	38+**95**	385	518
	1	36+**9**	42	87

Table: potential true table

		$\mathcal{Y}$			
		{a}	{b}	{a,b}	sum
UBII	0	38	385	**95**	518
	1	36	42	**9**	87

Table: observed table

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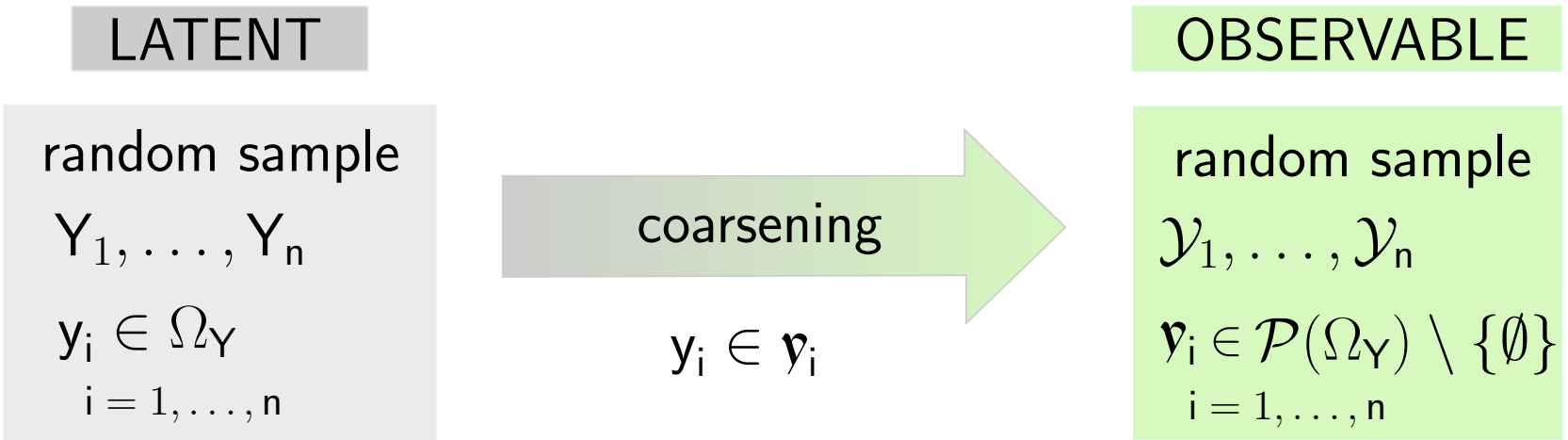

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	1	36	42+**9**	87

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		{a}	{b}	{a,b}	sum
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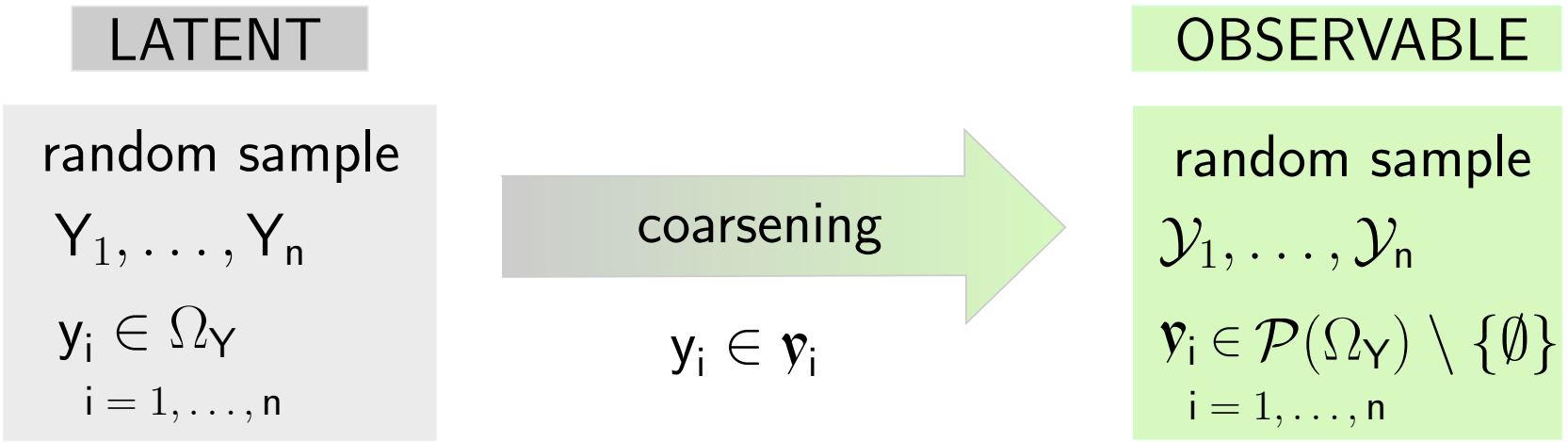

		$\mathcal{Y}$		
		a	b	sum
UBII	0	38+**20**	385+**75**	518
	1	36+**5**	42+**4**	87

Table: potential true table

		$\mathcal{Y}$			
		{a}	{b}	{a,b}	sum
UBII	0	38	385	**95**	518
	1	36	42	**9**	87

Table: observed table

General ML estimation (Plass, Augustin, Cattaneo, Schollmeyer, 2015)

LATENT

$\pi_{xy} :=$

$P(Y_i = y | X_i = x)$

(error-freeness)

Observation model

$q_{\mathcal{Y}|xy} :=$

$P(\mathcal{Y}_i = y | X_i = x, Y_i = y)$

OBSERVABLE

$p_{x\mathcal{Y}} :=$

$P(\mathcal{Y}_i = \mathcal{Y} | X_i = x)$

General ML estimation (Plass, Augustin, Cattaneo, Schollmeyer, 2015)

LATENT

$$\vartheta = (\pi_{xy}^T,\ q_{\mathcal{Y}|xy}^T)^T$$

OBSERVABLE

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General ML estimation (Plass, Augustin, Cattaneo, Schollmeyer, 2015)

LATENT

$$\vartheta = (\pi_{xy}^T,\ q_{y|xy}^T)^T$$

OBSERVABLE

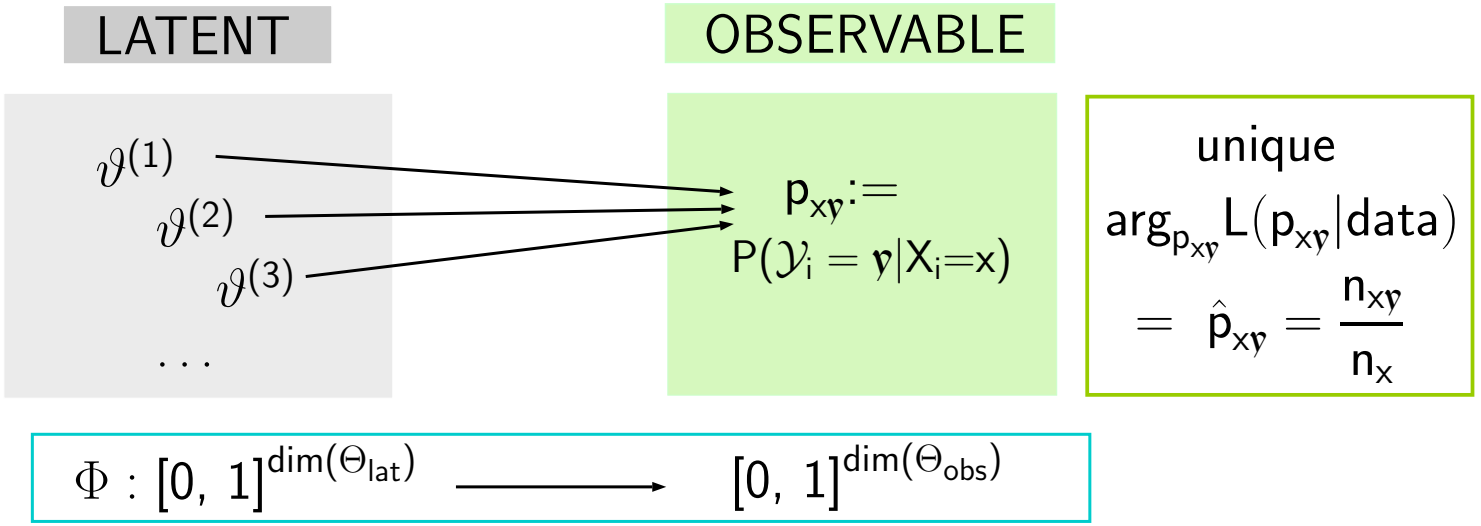
$$p_{xy} := P(\mathcal{Y}_i = y | X_i = x)$$

unique

$$\arg_{p_{xy}} L(p_{xy}|\text{data}) = \hat{p}_{xy} = \frac{n_{xy}}{n_x}$$

1.) Determine MLE of observed variable distribution

General ML estimation (Plass, Augustin, Cattaneo, Schollmeyer, 2015)



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2.) Use connection between both worlds

$$p_{x\mathcal{Y}} = \sum_{y \in \mathcal{Y}} \left(\pi_{xy} \cdot q_{\mathcal{Y}|xy} \right).$$

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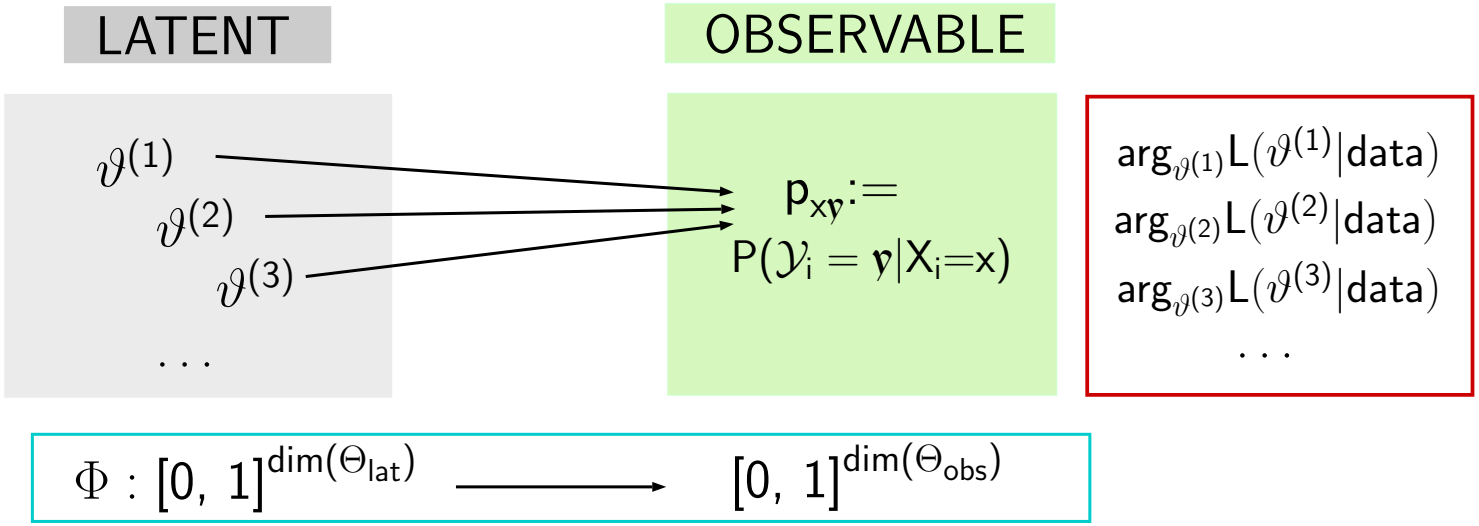

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$$p_{x\mathfrak{y}} = \sum_{y \in \mathfrak{y}} \left(\pi_{xy} \cdot q_{\mathfrak{y}|xy} \right).$$

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$$\hat{\pi}_{xy} \in \left[\frac{n_{x\{y\}}}{n_x}, \frac{\sum_{\mathfrak{y} \ni y} n_{x\mathfrak{y}}}{n_x} \right], \quad \hat{q}_{\mathfrak{y}|xy} \in \left[0, \frac{n_{x\mathfrak{y}}}{n_{x\{y\}} + n_{x\mathfrak{y}}} \right].$$

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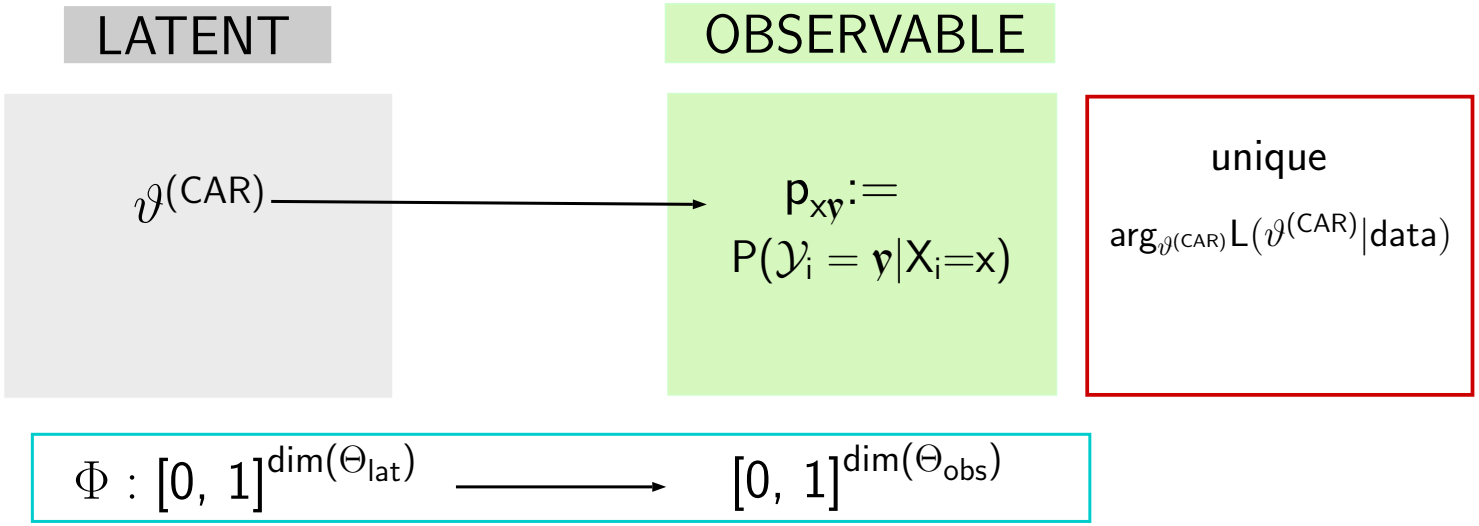

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Definition of CAR (Heitjan, Rubin, 1991):

For each fixed $\boldsymbol{y}$ and x, $q_{\boldsymbol{y}|xy}$ takes the same value $\forall y \in \boldsymbol{y}$.

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Illustrated by the example:

- The probability of {a,b} is taken to be independent of the true income category in both subgroups split by UBII:

  $q_{\{a,b\}|0\mathbf{a}} = q_{\{a,b\}|0\mathbf{b}}$ and $q_{\{a,b\}|1\mathbf{a}} = q_{\{a,b\}|1\mathbf{b}}$

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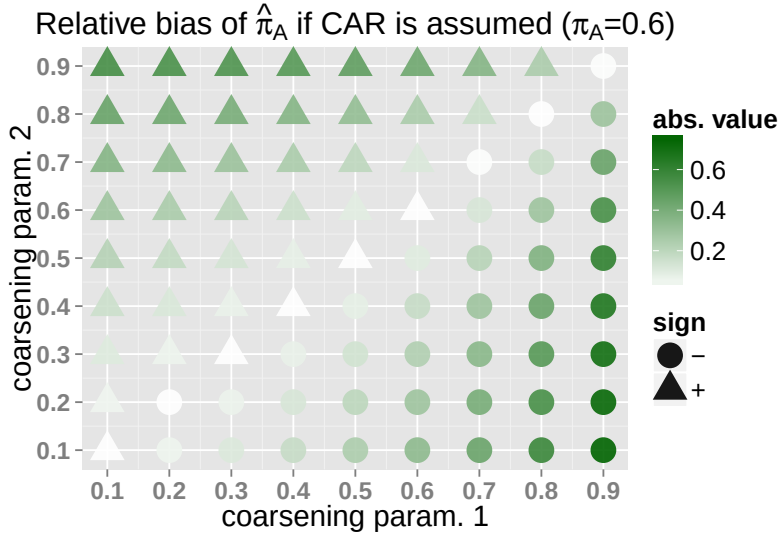

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- Resulting estimators:

$$\hat{\pi}_{xa}^{(CAR)} = \frac{n_{x\{a\}}}{n_{x\{a\}} + n_{x\{b\}}}, \quad \hat{q}_{\{a,b\}|xa}^{(CAR)} = \hat{q}_{\{a,b\}|xb}^{(CAR)} = \frac{n_{x\{a,b\}}}{n_x}$$

Testing CAR? $\Rightarrow$:-((e.g. Jaeger, 2006)

Estimators for subgroup 0:

$$\hat{\pi}_{0a} \in [0.07, 0.26], \quad \hat{q}_{\{a,b\}|0a} \in [0,\ 0.71], \quad \hat{q}_{\{a,b\}|0b} \in [0,\ 0.20]$$

$$\hat{\pi}_{0a}^{(CAR)} = 0.09, \qquad \hat{q}_{\{a,b\}|0a}^{(CAR)} = 0.18, \qquad \hat{q}_{\{a,b\}|0b}^{(CAR)} = 0.18$$

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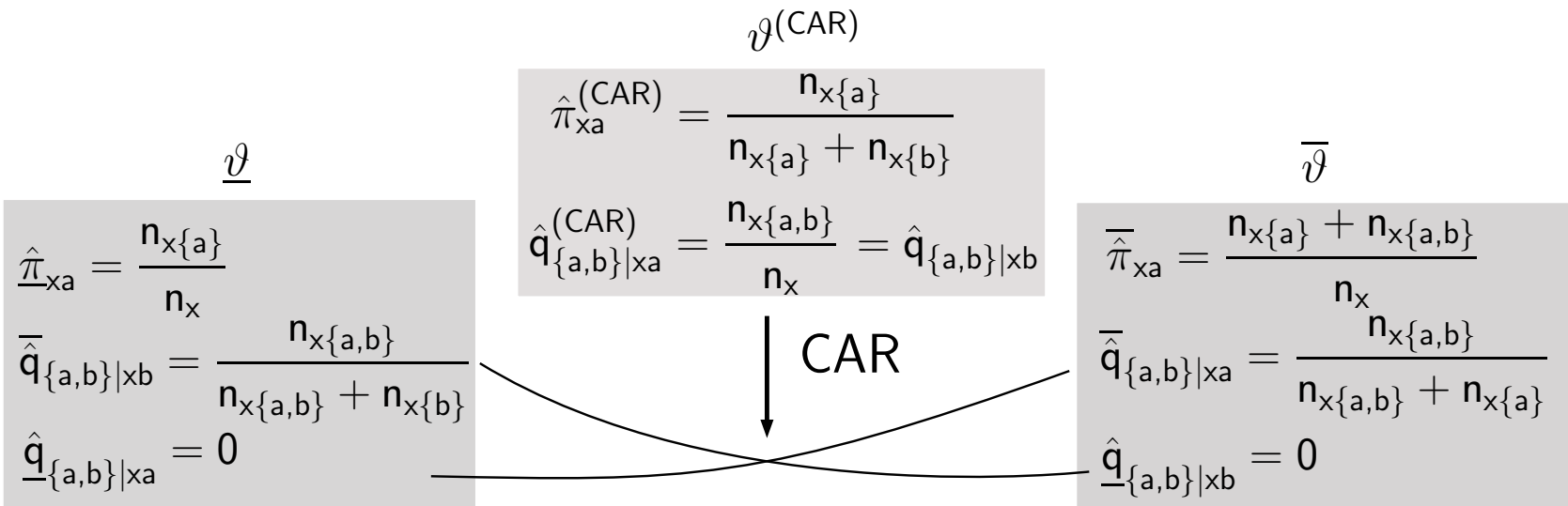

$\Rightarrow$ CAR is generally not testable, unless further assumptions about the coarsening are justified

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- Resulting estimators (if well-defined and inside [0,1]):

$$\hat{\pi}^{(SI)}_{xa} = \frac{n_{x\{a\}}}{n_x}\frac{v}{w}, \qquad \hat{q}^{(SI)}_{\{a,b\}|xa} = 1 - \frac{w}{v}, \quad \hat{q}^{(SI)}_{\{a,b\}|xb} = 1 - \frac{w}{z}$$

  with $v = n_0 n_{1\{b\}} - n_{0\{b\}} n_1$, $w = n_{0\{a\}} n_{1\{b\}} - n_{0\{b\}} n_{1\{a\}}$ and $z = n_{0A} n_1 - n_{1A} n_0$

Testing SI? $\Rightarrow$:-)

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$$\hat{\pi}_{0a}^{(SI)} = 0.42, \qquad \hat{q}_{\{a,b\}|0a}^{(SI)} = -0.04, \qquad \hat{q}_{\{a,b\}|0b}^{(SI)} = 0.20$$

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$\Rightarrow$ There are data situations that might hint to (partial) incompatibility with SI

$\Rightarrow$ SI is testable in our setting

Testing SI: Hypothesis and test statistic

- Hypothesis:

$$\begin{aligned} \mathsf{H}_0 &: \quad \mathsf{q}_{\mathfrak{y}|0\mathsf{y}} = q_{\mathfrak{y}|1\mathsf{y}} \text{ for all } \mathsf{y} \in \Omega_Y = \{\mathsf{a},\mathsf{b}\},\ \mathfrak{y} \in \mathcal{P}(\Omega_Y) \setminus \emptyset \\ \mathsf{H}_1 &: \quad \mathsf{q}_{\mathfrak{y}|0\mathsf{y}} \neq q_{\mathfrak{y}|1\mathsf{y}} \text{ for some } \mathsf{y} \in \Omega_Y = \{\mathsf{a},\mathsf{b}\},\ \mathfrak{y} \in \mathcal{P}(\Omega_Y) \setminus \emptyset. \end{aligned}$$

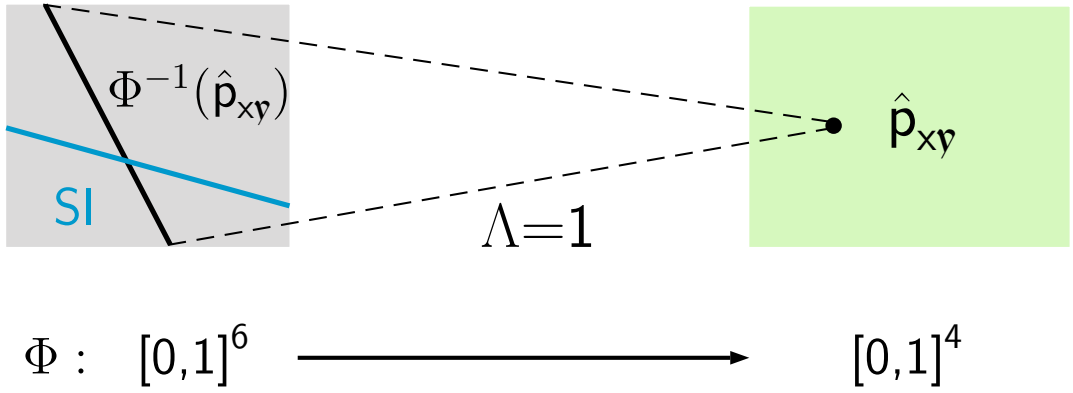
- Test based on test statistic:

$$\Lambda(\mathfrak{y}_1, \ldots, \mathfrak{y}_n, \mathsf{x}_1, \ldots, \mathsf{x}_\mathsf{n}) = \frac{\sup_{\mathsf{H}_0} L(\vartheta | \mathfrak{y}_1, \ldots, \mathfrak{y}_\mathsf{n}, \mathsf{x}_1, \ldots, \mathsf{x}_\mathsf{n})}{\sup_{\mathsf{H}_0 \cup \mathsf{H}_1} L(\vartheta | \mathfrak{y}_1, \ldots, \mathfrak{y}_\mathsf{n}, \mathsf{x}_1, \ldots, \mathsf{x}_\mathsf{n})},$$

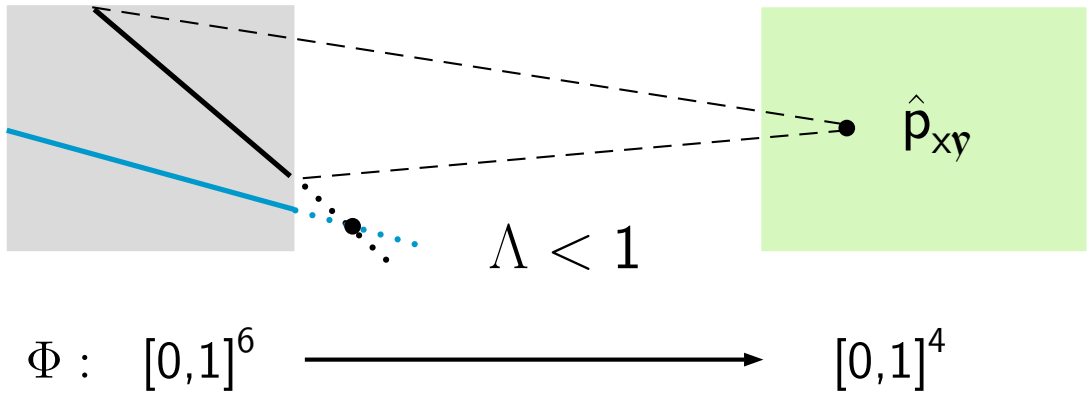
with $\vartheta = (\pi_{0\mathsf{a}}, \pi_{1\mathsf{a}}, \mathsf{q}_{\{\mathsf{a},\mathsf{b}\}|0\mathsf{a}}, \mathsf{q}_{\{\mathsf{a},\mathsf{b}\}|1\mathsf{a}}, \mathsf{q}_{\{\mathsf{a},\mathsf{b}\}|0\mathsf{b}}, \mathsf{q}_{\{\mathsf{a},\mathsf{b}\}|1\mathsf{b}})^T$

Testing SI: Sensitivity of Λ

- No evidence to reject SI



- Some evidence to reject SI



- Result of the data example:
  $\Lambda \approx 0.93 \Rightarrow$ slight evidence against SI

Studying more general settings...

Now: number of subgroups m, $|\Omega_Y| = k$

CAR is generally known to be ...

- ... point-identifying
  (Heitjan, Rubin, 1991)
- ... not testable
  (e.g. Jaeger, 2006)

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CAR is generally known to be ...

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SI: A determination of the number of degrees of freedom df is crucial:

$$\text{df} = \dim(\Theta_{\text{obs}}) - \dim(\Theta_{\text{lat, SI}}) = 2^{\text{m-1}}(2\text{k-m})\text{-}(\text{m}+1)(\text{k-1})\text{-}1$$

- Point-identification and testability are only valid if sufficient subgroups are available inducing $\text{df} \geq 0$
- If $\text{df} \geq 0$, a LR-Test on SI may be constructed with ...
  - ... test statistic $T = -2 \cdot \log(\Lambda)$ (Wilks, 1938)
  - ... asymptotic distribution of T under H_0
    - $\frac{1}{2}\delta_0 + \frac{1}{2}\chi^2_1$ if df=0
    - χ^2_{df} if df>0,

    where δ_0 is the Dirac distribution at 0 (Chernoff, 1954)

Summary: CAR versus SI

	CAR	SI
Point-identifying?	always	in specific settings
Testability	generally impossible	possible in specific settings

Construction of a hypothesis test:

- H_0: SI, H_1: no SI
- Test statistic based on the Likelihood Ratio

Next step: Generalized version of LR-Test on SI

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